

Problem 1

Prove that an element e in a matroid is contained in all bases if and only if it is not contained in any circuit.

Forward Implication. Suppose the element e is contained in all the bases. If it were contained in some circuit, we could remove it from that circuit and get an independent set A . Let $|C|$ denote the cardinality of the bases. If $|A| = |C|$, A must be a basis which does not contain the element e and hence contradicts the fact that e is contained in all the bases. If $|A| < |C|$, there must be a basis $B \supset A$ as A is not maximally independent. $e \notin B$ because $A \cup e$ is a circuit. But this contradicts the fact that e is contained in all the bases. Hence the hypothesis that e is contained in some circuit is false. Therefore, if an element is contained in all the bases, then it is not contained in any circuit.

no need
to consider separate cases

Okay!

Backward Implication. Suppose the element e is not contained in any circuit. If there were a basis B such that $e \notin B$, then we could add e to B to get a dependent set. $B \cup e$ would then contain a circuit. The circuit cannot be completely contained in B as B is an independent set, and hence the circuit would contain the element e . But this contradicts the fact that e is not contained in any circuit. Hence our hypothesis that a basis contains e is false. Therefore, if an element is not contained in any circuit, then it is contained in all the bases.

Problem 2

Let B be a collection of subsets of a finite set E , satisfying the following properties:

- B is not empty.
- For any two sets B_1, B_2 in B , and any element e in B_1 , there exists an element f in B_2 such that $(B_1 - e) \cup f$ is in B .

Give an example to show that B need not be the set of bases of a matroid.

Suppose, we add another condition:

- No subset in B is a proper subset of another subset in B .

Now show that B must be the set of bases of some matroid.

Take $B = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{3, 1\}, \{1, 2, 3\}\}$. The collection B clearly satisfies the first two properties but the sets it contains cannot all be bases as they have different cardinalities.

okay!

Take two sets B_1 and B_2 in B . B_1 is not a proper subset of B_2 by the third property. By second property, for any element $e \in (B_1 - B_2)$, $\exists f \in B_2$ such that $(B_1 - e) \cup f$ is in B . If $f \in (B_1 \cap B_2)$, $((B_1 - e) \cup f) \subset B_1$, but this is not allowed by third property. Hence $f \in (B_2 - B_1)$. Therefore, the collection B satisfies the two properties:

- B is not empty.
- If B_1 and B_2 are distinct subsets in B , then for every $x \in (B_1 - B_2)$, $\exists y \in (B_2 - B_1)$ such that $(B_1 - x) \cup y$ is in B .

okay!

Hence, B must be the set of bases of some matroid.

Problem 3

Give an example of two matroids M_1 and M_2 on the same set E , such that a subset X of E is a circuit of M_1 iff X is a basis of M_2 . Characterize all possible such pairs of matroids.

For $m + 1 \leq n$, take $M_1 = U_{n,m}$ and $M_2 = U_{n,m+1}$ with the same ground set E . The $m + 1$ element subsets of E are exactly all the circuits in M_1 and all the bases in M_2 .

Okay, are these the only examples?

Problem 4

Let M be a matroid and X and Y be subsets of $E(M)$, each of which contains a basis. Prove that if $|X| > |Y|$ then for some element x in $X - Y$, $X - x$ also contains a basis.

Take the bases B_1 and B_2 such that $B_1 \subset X$ and $B_2 \subseteq Y$. If there is an element $x \in (X - Y)$ such that $x \notin B_1$, then $X - x$ contains the basis B_1 . Else, $X - Y \subset B_1$ and hence $|B_1 \cap (X - Y)| = |X - Y|$. $|X| > |Y| \Rightarrow |X - Y| > |Y - X| \Rightarrow |B_1 \cap (X - Y)| > |B_2 \cap (Y - X)| \Rightarrow |B_2 \cap (X \cap Y)| > |B_1 \cap (X \cap Y)|$. Construct basis B_3 as follows: start with B_2 and exchange the elements in $B_2 \cap (Y - X)$ with elements in $B_1 - B_2$. $|B_3 \cap (X \cap Y)| \geq |B_2 \cap (X \cap Y)| > |B_1 \cap (X \cap Y)|$. Therefore, $|B_3 \cap (X - Y)| < |B_1 \cap (X - Y)| = |X - Y|$. So, $\exists x \in (X - Y) - B_3$ and hence $X - x$ contains the basis B_3 .

okay, simpler argument possible.

Problem 5

Show that if B_1, B_2 are any two distinct bases of a matroid, then for every element e in $B_2 - B_1$, there exists an element f in $B_1 - B_2$, such that $B_1 + e - f$ is also a basis. Show that any non-empty collection of subsets satisfying the property must be the set of bases of a matroid.

For every e in $B_2 - B_1$, $(B_1 \cap B_2) \cup \{e\}$ is an independent set, since it is a subset of B_2 . Take $B_3 = (B_1 \cap B_2) \cup \{e\}$. While $|B_3| < |B_1|$, we can add elements from $B_1 - B_3 \subseteq B_1 - B_2$ to B_3 while keeping B_3 independent. Finally $|B_3| = |B_1|$, and so B_3 is a basis. Also, $|(B_3 \cap (B_1 - B_2)) \cup \{e\}| = |B_1 - B_2| \Rightarrow |B_3 \cap (B_1 - B_2)| + 1 = |B_1 - B_2|$. Therefore there exists an element $f \in B_1 - B_2$ not contained in B_3 , and $B_3 = (B_1 - \{f\}) \cup \{e\}$.

okay!

Now suppose B is a non-empty set which satisfies the given property. We wish to prove that if B_1 and B_2 are distinct subsets in B , then for every $e \in (B_1 - B_2)$, $\exists f \in (B_2 - B_1)$ such that $(B_1 - e) \cup f$ is in B . Suppose the statement were false, then for some e in $B_1 - B_2$, $(B_1 - e) \cup f$ would not be in B for any $f \in B_2 - B_1$. Construct B_3 as follows: start with $B_3 = B_2$. From the set property, for every element $x \in B_1 - B_3$ we can replace an element of $B_3 - B_1$ with x , keeping B_3 in B . Do the replacements for every element in $B_1 - B_3$ except e . Finally we obtain $B_3 = (B_1 - e) \cup f$ for some $f \in B_2$. But this contradicts the presumption that $(B_1 - e) \cup f$ is not in B . Hence the statement we wished to prove is true, and this implies that B is a basis set for some matroid.

okay!

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