

Problem 1

Prove that an element e in a matroid is contained in all bases if and only if it is not contained in any circuit.

Forward Implication. Suppose the element e is contained in all the bases. If it were contained in some circuit, we could remove it from that circuit and get an independent set A . Let $|C|$ denote the cardinality of the bases. If $|A| = |C|$, A must be a basis which does not contain the element e and hence contradicts the fact that e is contained in all the bases. If $|A| < |C|$, there must be a basis $B \supset A$ as A is not maximally independent. $e \notin B$ because $A \cup e$ is a circuit. But this contradicts the fact that e is contained in all the bases. Hence the hypothesis that e is contained in some circuit is false. Therefore, if an element is contained in all the bases, then it is not contained in any circuit.

Backward Implication. Suppose the element e is not contained in any circuit. If there were a basis B such that $e \notin B$, then we could add e to B to get a dependent set. $B \cup e$ would then contain a circuit. The circuit cannot be completely contained in B as B is an independent set, and hence the circuit would contain the element e . But this contradicts the fact that e is not contained in any circuit. Hence our hypothesis that a basis contains e is false. Therefore, if an element is not contained in any circuit, then it is contained in all the bases.

Problem 2

Let B be a collection of subsets of a finite set E , satisfying the following properties:

- B is not empty.
- For any two sets B_1, B_2 in B , and any element e in B_1 , there exists an element f in B_2 such that $(B_1 - e) \cup f$ is in B .

Give an example to show that B need not be the set of bases of a matroid.

Suppose, we add another condition:

- No subset in B is a proper subset of another subset in B .

Now show that B must be the set of bases of some matroid.

Take $B = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{3, 1\}, \{1, 2, 3\}\}$. The collection B clearly satisfies the first two properties but the sets it contains cannot all be bases as they have different cardinalities.

Take two sets B_1 and B_2 in B . B_1 is not a proper subset of B_2 by the third property. By second property, for any element $e \in (B_1 - B_2)$, $\exists f \in B_2$ such that $(B_1 - e) \cup f$ is in B . If $f \in (B_1 \cap B_2)$, $((B_1 - e) \cup f) \subset B_1$, but this is not allowed by third property. Hence $f \in (B_2 - B_1)$. Therefore, the collection B satisfies the two properties:

- B is not empty.
- If B_1 and B_2 are distinct subsets in B , then for every $x \in (B_1 - B_2)$, $\exists y \in (B_2 - B_1)$ such that $(B_1 - x) \cup y$ is in B .

Hence, B must be the set of bases of some matroid.

Problem 3

Give an example of two matroids M_1 and M_2 on the same set E , such that a subset X of E is a circuit of M_1 iff X is a basis of M_2 . Characterize all possible such pairs of matroids.

For $m + 1 \leq n$, take $M_1 = U_{n,m}$ and $M_2 = U_{n,m+1}$ with the same ground set E . The $m + 1$ element subsets of E are exactly all the circuits in M_1 and all the bases in M_2 .

Problem 4

Let M be a matroid and X and Y be subsets of $E(M)$, each of which contains a basis. Prove that if $|X| > |Y|$ then for some element x in $X - Y$, $X - x$ also contains a basis.

Take the bases B_1 and B_2 such that $B_1 \subset X$ and $B_2 \subseteq Y$. If there is an element $x \in (X - Y)$ such that $x \notin B_1$, then $X - x$ contains the basis B_1 . Else, $X - Y \subset B_1$ and hence $|B_1 \cap (X - Y)| = |X - Y|$. $|X| > |Y| \Rightarrow |X - Y| > |Y - X| \Rightarrow |B_1 \cap (X - Y)| > |B_2 \cap (Y - X)| \Rightarrow |B_2 \cap (X \cap Y)| > |B_1 \cap (X \cap Y)|$. Construct basis B_3 as follows: start with B_2 and exchange the elements in $B_2 \cap (Y - X)$ with elements in $B_1 - B_2$. $|B_3 \cap (X \cap Y)| \geq |B_2 \cap (X \cap Y)| > |B_1 \cap (X \cap Y)|$. Therefore, $|B_3 \cap (X - Y)| < |B_1 \cap (X - Y)| = |X - Y|$. So, $\exists x \in (X - Y) - B_3$ and hence $X - x$ contains the basis B_3 .

Problem 5

Show that if B_1, B_2 are any two distinct bases of a matroid, then for every element e in $B_2 - B_1$, there exists an element f in $B_1 - B_2$, such that $B_1 + e - f$ is also a basis. Show that any non-empty collection of subsets satisfying the property must be the set of bases of a matroid.

For every e in $B_2 - B_1$, $(B_1 \cap B_2) \cup \{e\}$ is an independent set, since it is a subset of B_2 . Take $B_3 = (B_1 \cap B_2) \cup \{e\}$. While $|B_3| < |B_1|$, we can add elements from $B_1 - B_3 \subseteq B_1 - B_2$ to B_3 while keeping B_3 independent. Finally $|B_3| = |B_1|$, and so B_3 is a basis. Also, $|(B_3 \cap (B_1 - B_2)) \cup \{e\}| = |B_1 - B_2| \Rightarrow |B_3 \cap (B_1 - B_2)| + 1 = |B_1 - B_2|$. Therefore there exists an element $f \in B_1 - B_2$ not contained in B_3 , and $B_3 = (B_1 - \{f\}) \cup \{e\}$.

Now suppose B is a non-empty set which satisfies the given property. We wish to prove that if B_1 and B_2 are distinct subsets in B , then for every $e \in (B_1 - B_2)$, $\exists f \in (B_2 - B_1)$ such that $(B_1 - e) \cup f$ is in B . Suppose the statement were false, then for some e in $B_1 - B_2$, $(B_1 - e) \cup f$ would not be in B for any $f \in B_2 - B_1$. Construct B_3 as follows: start with $B_3 = B_2$. From the set property, for every element $x \in B_1 - B_3$ we can replace an element of $B_3 - B_1$ with x , keeping B_3 in B . Do the replacements for every element in $B_1 - B_3$ except e . Finally we obtain $B_3 = (B_1 - e) \cup f$ for some $f \in B_2$. But this contradicts the presumption that $(B_1 - e) \cup f$ is not in B . Hence the statement we wished to prove is true, and this implies that B is a basis set for some matroid.