

Connectivity

Problem 1

Let M be a matroid and X a subset of $E(M)$ such that $M|X$ is connected. Find necessary and sufficient conditions for $M|cl(X)$ to be connected.

X is contained in a connected component of M . $M|cl(X)$ will not be connected if there exists an element $r(X \cup y) = r(X)$, which lies in a different component. Let A be the largest independent subset in X . Since $y \cup A$ is independent if y is independent and in another connected component, y can belong to the closure if and only if it is a loop. Therefore, necessary and sufficient condition for $M|cl(X)$ to be connected is that there shouldn't be any loops in the matroid.

Problem 2

Let M be a connected matroid such that $M \setminus e$ is disconnected for all elements e in $E(M)$. Prove that M has a cocircuit of size 2. Also show that every circuit in M is a flat, that $cl(C) = C$ for any circuit C . Suppose M is an n -connected matroid such that $M \setminus e$ is not n -connected for all elements e . Does M necessarily contain a cocircuit of size n ?

Since M is connected there must be at-least one circuit C in the matroid. We then use the theorem that if $M \setminus e_i$ is disconnected for all $e_i \in C$, then there exists a co-circuit of size 2 in the circuit.

Let x be an element outside C . We show that $x \cup C$ does not contain a circuit, else x would be in the closure of C . Suppose it did contain a circuit C_1 . Then any 2 elements e, f which were connected through a circuit C_2 containing x would still be connected. To check this, observe that there exists a circuit C_3 in $C_1 \cup C_2 - x$ containing e . Similarly, there exists a circuit C_4 in $C_1 \cup C_2 - x$ containing f . Now e is connected to some $c_1 \in C$ through C_3 , c_1 through C is connected to c_2 , which is connected to f through C_4 . Hence e is connected to f through circuits not involving x . Since this holds for all e, f , deleting x leaves the graph connected, which contradicts the property that deletion of any element disconnects the matroid.

Problem 3

Let M be a connected matroid with at least two elements and let e be an arbitrary element. Suppose the largest circuit containing e has l elements. Prove that there exist $l - 1$ cocircuits, each containing e , whose union is $E(M)$.

Problem 4

Suppose that x and y are distinct elements of an n -connected matroid M , where $n \geq 2$ and $|E(M)| \geq 2(n - 1)$. Suppose that $M \setminus x/y$ is n -connected but $M \setminus x$ is not. Prove that M has a cocircuit of size n containing x and y .

Problem 5

Let M be the 2-sum of two connected matroids M_1 and M_2 . Prove that there exists a minor of M isomorphic to M_1 and a minor of M isomorphic to M_2 . Let N be a 3-connected minor of M . Prove that N is a minor of either M_1 or M_2 .

Let E_1 be the edge set in M derived from M_1 , similarly E_2 from M_2 . There exists a circuit C with elements in both E_1 and E_2 . Delete $E_2 - C$ from M and contract all but one elements of $C - E_1$. Circuits are preserved under contraction and it is easy to check from the 2-sum construction (via circuits) that the circuits in the obtained matroid are same as circuits of M_1 . The matroid obtained is isomorphic to M_1 .

M is the 2-sum of M_1 and M_2 . There is a 2-separation between M_1 and M_2

$$\lambda_M(X) = r_M(E_1) + r_M(E_2) - r(M) + 1$$

If $r(M)$ decreases then either $r_M(E_1)$ or $r_M(E_2)$ would also decrease as independent sets in M are unions of independent sets from M_1 and M_2 (subtracting p). Therefore, connectivity between E_1 and E_2 would not increase upon contraction or deletion. Therefore, if N has to be 3-connected, it must be restricted to either M_1 or M_2 in which case it will be a minor of the either of the two.