

Problem

What is the maximum number of elements in a rank- r binary matroid having no two disjoint circuits?

Consider the circuit space of a rank- r binary matroid having no two disjoint circuits. Since the XOR of two circuits in a binary matroid is a disjoint union of circuits, and there are no two disjoint circuits, the XOR must itself be a circuit. Since the rank of the matroid is r , there exists an r -element subset of the ground set not containing any circuit. Label this subset E , $|E| = r$. Label the elements outside the set as $c_1, c_2, \dots, c_{n-r}$, where n is the cardinality of the ground set G . However, every $(r+1)$ -element subset of ground set must contain a circuit. And hence there must be a circuit in $E \cup c_1$. Label this circuit C_1 . There cannot be any other circuit in $E \cup c_1$ else either that circuit or its XOR with C_1 would be in E . Similarly we obtain circuits $C_2, C_3 \dots C_{n-r}$ for each of the subsets $E \cup c_2, E \cup c_3, \dots, E \cup c_{n-r}$. Now, if a circuit C has $C \cap E^c = \cup_{i \in \Lambda} \{c_i\}$ for some subset Λ of $N = \{1, \dots, n-r\}$, it must be $\bigoplus_{i \in \Lambda} C_i$, else its XOR with $\bigoplus_{i \in \Lambda} C_i$ will be a subset contained in E , which cannot not be a circuit.

Hence we see that any circuit C can be written as $\bigoplus_{i \in \Lambda} C_i$ for some subset Λ of $N = \{1, \dots, n-r\}$. Furthermore, the following properties have to be satisfied

- Now $|\Lambda|$ is the number of elements in $C \cap E^c$, and hence the maximum number of elements in $C \cap E$ can be $r+1-|\Lambda|$.
- Also since no circuit can be a proper subset of other, we must have that $\bigoplus_{i \in \lambda} C_i$ must not be a subset of $\bigoplus_{i \in \Lambda} C_i$ for $\lambda \subset \Lambda$. This is equivalent to condition that $E \cap \bigoplus_{i \in \lambda} C_i$ not be a subset of $E \cap \bigoplus_{i \in \Lambda} C_i$. This is also equivalent to condition that $E \cap \bigoplus_{i \in \lambda} C_i$ should not be disjoint from $E \cap \bigoplus_{i \in \Lambda^c} C_i$.
- Every $(r+1)$ -subset must have a circuit contained in it. Hence, for every $(r+1)$ subset S of G , there exists a circuit $C = \bigoplus_{i \in \lambda} C_i$, $\lambda \subseteq \Lambda$ where $\cup_{i \in \lambda} \{c_i\} = S \cap E^c$, such that $E \cap \bigoplus_{i \in \lambda} C_i \subseteq S \cap E^c$.

From the condition that number of elements in C has to be less than or equal to $r+1-|\Lambda|$, we get a naive upper bound, $r+1-(n-r) \geq 0 \Rightarrow n \leq 2r+1$.

A better bound can be obtained as follows. Take any subset A of E^c such that $|A| = a$ and $A = \cup_{i \in \lambda} c_i$, then for every $r+1-a$ subset in E , its union with A should contain a circuit. From the second property, $C = \bigoplus_{i \in N-\lambda} C_i$ must intersect with every $r+1-a$ subset in E , and hence must have at-least $r-(r-a) = a$ elements from E in addition to $p-a \equiv (n-r)-a = |N-\lambda|$ elements from E^c . These combined must be less than or equal to $r+1$.